Threshold Secret Sharing: Information-Theoretic

March 8, 2020

Threshold Secret Sharing

Zelda has a **secret** $s \in \{0,1\}^n$.

Def: Let $1 \le t \le m$. (t, m)**-secret sharing** is a way for Zelda to give strings to $A_1, \ldots, A_m$ such that:

1. If any t get together than they can learn s
2. If any $t-1$ get together they cannot learn s

What do we mean by **Cannot learn the secret**? We mean info-theory-security. Even if $t-1$ people have big fancy supercomputers they cannot learn s. We will later look at comp-security.

Applications

Rumor: Secret Sharing is used for the Russian Nuclear Codes. There are three people (one is Putin) and if two of them agree to launch, they can launch.

For people signing a contract long distance, secret sharing is used as a building block in the protocol.

$(4,4)$-secret sharing

Zelda has a secret s. A_1, A_2, A_3, A_4 are people. We want:

1. If all four of A_1, A_2, A_3, A_4 get together, they can find s.
2. If any three of them get together, then learn **NOTHING**.

An Attempt at $(4, 4)$-Secret Sharing

1. Zelda breaks s up into $s = s_1 s_2 s_3 s_4$ where

$$|s_1| = |s_2| = |s_3| = |s_4| = \frac{n}{4}$$

2. Zelda gives A_i the string s_i.

Does this work?

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1. If A_1, A_2, A_3, A_4 get together they can find s. **YES!!**
2. If any three of them get together they learn **NOTHING**. **NO**.
    2.1 A_1 learns s_1 which is $\frac{1}{4}$ **of the secret!**
    2.2 A_1, A_2 learn $s_1 s_2$ which is $\frac{1}{2}$ **of the secret!**
    2.3 A_1, A_2, A_3 learn $s_1 s_2 s_3$ which is $\frac{3}{4}$ **of the secret!**

What do we mean by **NOTHING**?

If any three of them get together they learn **NOTHING**
Informally:

1. Before Zelda gives out shares, if any three A_i, A_j, A_k get together, they know $BLAH_{i,j,k}$.
2. After Zelda gives out shares, if any three A_i, A_j, A_k get together, they know $BLAH_{i,j,k}$. (This is the same $BLAH_{i,j,k}$ as in the first point.
3. Giving out the shares tells A_1, A_2, A_3, A_4 **NOTHING** that they did not already know.

We assume A_i, A_j, A_k have **unlimited computing power**.

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Information-Theoretic Security

Is $(4,4)$-Secret Sharing Possible?

VOTE: Is $(4,4)$-Secret sharing possible?

1. YES
2. NO
3. YES given some hardness assumption
4. UNKNOWN TO SCIENCE

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   Zelda gives A_2 $s_2 = r_2$.
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   Zelda gives A_4 $s_4 = s \oplus r_1 \oplus r_2 \oplus r_3$

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A_1, A_2, A_3 A_4 Can Recover the Secret

$$s_1 \oplus s_2 \oplus s_3 \oplus s_4 = r_1 \oplus r_2 \oplus r_3 \oplus r_1 \oplus r_2 \oplus r_3 \oplus s = s$$

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Easy to see that if a 3 get together they learn NOTHING

$(2, 4)$-Secret Sharing using Random Strings-Intuitive

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Do same for (A_1, A_4), (A_2, A_3), (A_3, A_4).
Question Is there a problem with this?

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Zelda gives A_1 r (to use when talking to A_2)
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Zelda needs to give A_1 strings of the form
$((1,j), r)$: This is a string to be used when A_1 and A_j are talking.
Caveat Don't need to tell A_1 who he is, but notation will generalize.

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A_i, A_j Can Recover the Secret

A_i takes $((i,j), r)$ and just uses the r.
A_j takes $((i,j), r \oplus s)$ and just uses the $r \oplus s$.
They both compute $r \oplus r \oplus s = s$.

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Easy to see that one person learns NOTHING

(m, m)-Random String Method

People: $A_1, \ldots, A_m$. Secret s.

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We use this as building block for gen case.

(t, m) Secret Sharing

People: $A_1, \ldots, A_m$. $S_1, \ldots, S_{\binom{m}{t}} \subseteq \{A_1, \ldots, A_m\}$ are t-subsets.

1. For every $1 \le j \le \binom{m}{t}$ Zelda does (t, t) secret sharing with the elements of S_j but also prepends every string with j.
2. If the people in S_j get together they XOR together strings prepended with j (do not use the j).
3. No smaller subset can get the secret.

PRO: Can always do Threshold Secret Sharing.
CON: You are giving people A LOT of strings!

A_i Gets ??? Strings in $(5, 10)$-Secret Sharing

If do $(5, 10)$ secret sharing then how many strings does A_1 get?

A_1 gets a string for every $J \subseteq \{1, \ldots, 10\}$, $|J| = 5$, $1 \in J$.

Equivalent to:

A_1 gets a string for every $J \subseteq \{2, \ldots, 10\}$, $|J| = 4$.

How many sets? **Discuss**

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$$\binom{9}{4} = 126 \text{ strings}$$

A_i Gets ??? Strings in $(m/2, m)$-Secret Sharing

If do $(m/2, m)$ secret sharing then how many strings does A_1 get?

A_1 gets a string for every $J \subseteq \{1, \ldots, m\}$, $|J| = \frac{m}{2}$, $1 \in J$.
Equivalent to:

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How many sets? **Discuss**

$$\binom{m-1}{\frac{m}{2}-1} \sim \frac{2^m}{\sqrt{m}} \text{ strings}$$

Thats A LOT of Strings!

Reduce The Number of Strings for $(m/2, m)$?

In our $(m/2, m)$-scheme each A_i gets $\sim \frac{2^m}{\sqrt{m}}$ strings.

VOTE

1. Requires roughly 2^m strings.
2. $O(\beta^m)$ strings for some $1 < \beta < 2$ but not poly.
3. $O(m^a)$ strings for some $a > 1$ but not linear.
4. $O(m)$ strings but not m^a with $a < 1$.
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You can always do this with everyone getting 1 string

I know what you are thinking:LOOOONG string.No.

You can always do this with everyone getting 1 string that is the same length as the secret

Secret Sharing With Polynomials

Definition $a \sim b$ means $\frac{b}{2} \le a \le 2b$.
We do $(3, 6)$-Secret Sharing.

1. Secret s. Zelda picks prime $p \sim 2^{|s|}$, Zelda works mod p. View s as a number is in $\{0, \ldots, p-1\}$.
2. Zelda gen rand numbers $a_2, a_1 \in \{0, \ldots, p-1\}$
3. Zelda forms polynomial $f(x) = a_2x^2 + a_1x + s$.
4. Zelda gives A_1 $f(1)$, A_2 $f(2)$, ..., A_6 $f(6)$ (all mod p). These are all of length $|s|$ by padding with 0's. Also give everyone p (does not count for length).

1. Any 3 have 3 points from $f(x)$ so can find $f(x)$, s.
2. Any 2 have 2 points from $f(x)$. From these two points what can they conclude?

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1. Any 3 have 3 points from $f(x)$ so can find $f(x)$, s.
2. Any 2 have 2 points from $f(x)$. From these two points what can they conclude? NOTHING! If they know $f(1) = 3$ and $f(2) = 7$ and f is degree 2 then the constant term can be **anything** in $\{0, \ldots, p\}$. So they know NOTHING about s.

What Counts

We are concerned about the size of the shares.

1. If Zelda **broadcasts to everyone** a string p, that is not counted towards someone share.
2. If Zelda gives A_1 a string that nobody else gets then that is A_1's share and that counts.
3. If Zelda gives A_1 and A_2 a string (and they both know its the same string) but nobody else, should that count as the length of the share?

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Example

$s = 10100$. We'll use $p = 23$.
(ADDED LATER- TAKING $P = 23$ IS IS INCORRECT!! WILL REVIST THIS POINT IN THIRD SET OF SLIDES ON SEC SHARING.)

1. Zelda picks $a_2 = 8$ and $a_1 = 13$.
2. Zelda forms polynomial $f(x) = 8x^2 + 13x + 20$.
3. Zelda gives A_1 $f(1) = 18$, A_2 $f(2) = 9$, A_3 $f(3) = 16$, A_4 $f(4) = 16$, A_5 $f(5) = 9$, A_6 $f(6) = 18$.

If A_1, A_3, A_4 get together and want to find $f(x)$ hence s.
$f(x) = a_2x^2 + a_1x + s$.
$f(1) = 18$: $a_2 \times 1^2 + a_1 \times 1 + s \equiv 18 \pmod{23}$
$f(3) = 16$: $a_2 \times 3^2 + a_1 \times 3 + s \equiv 16 \pmod{23}$
$f(4) = 16$: $a_2 \times 4^2 + a_1 \times 4 + s \equiv 16 \pmod{23}$
3 linear equations in, 3 variable, over mod 23 can be solved.
Note: Only need constant term s but can get all coeffs.

What if Two Get Together?

What if A_1 and A_3 get together:
$f(1) = 18$: $a_2 \times 1^2 + a_1 \times 1 + s \equiv 18 \pmod{23}$
$f(3) = 16$: $a_2 \times 3^2 + a_1 \times 3 + s \equiv 16 \pmod{23}$
Can they solve these to find s **Discuss**.

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$f(1) = 18$: $a_2 \times 1^2 + a_1 \times 1 + s \equiv 18 \pmod{23}$
$f(3) = 16$: $a_2 \times 3^2 + a_1 \times 3 + s \equiv 16 \pmod{23}$
Can they solve these to find s **Discuss**.

No. However, can they use these equations to eliminate some values of s? **Discuss**.

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Important: Information-Theoretic Secure: if A_1 and A_3 meet they learn NOTHING. If they had big fancy supercomputers they would still learn NOTHING.

A Note About Linear Equations

The three equations below, over mod 23, can be solved:

$a_2 \times 1^2 + a_1 \times 1 + s \equiv 18 \pmod{23}$
$a_2 \times 3^2 + a_1 \times 3 + s \equiv 16 \pmod{23}$
$a_2 \times 4^2 + a_1 \times 4 + s \equiv 16 \pmod{23}$

Could we have solved this had we used mod 24?
VOTE

1. YES
2. NO

A Note About Linear Equations

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Could we have solved this had we used mod 24?

VOTE

1. YES
2. NO

These equations, Don't know, but in general, NO

Need a domain where every number has a mult inverse.

Over mod p, p primes, all numbers have mult inverses.

Over mod 24, even numbers do not have mult inverse.

Subtle Point about Length p

You may have noticed the following oddness:

1. I said **pick** $p \sim 2^{|s|}$.
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Let $s \in \{0,1\}^n$. So how to best pick prime p?

1. Need prime p such that the string s **interpreted as a number in binary** is in $\{0, \ldots, p-1\}$.
2. Want smallest such prime p.
3. p a prime $\geq 2^{|s|}$ always works.
4. Often can use a smaller prime.
5. $s = 10100$. Need a prime such that $20 \in \{0, \ldots, p-1\}$. $p = 23$ is smallest.
6. $s = 11111$. Need a prime such that $31 \in \{0, \ldots, p-1\}$. $p = 37$ is smallest.

Threshold Secret Sharing With Polynomials: Ref

Due to Adi Shamir

How to Share a Secret

Communication of the ACM

Volume 22, Number 11

1979

Threshold Secret Sharing With Polynomials

Zelda wants to give strings to $A_1, \ldots, A_m$ such that

Any t of $A_1, \ldots, A_m$ can find s. Any $t-1$ learn **NOTHING**.

1. Secret s. Zelda picks prime $p \sim 2^{|s|}$, Zelda works mod p.
2. Zelda gen rand $a_{t-1}, \ldots, a_1 \in \{0, \ldots, p-1\}$
3. Zelda forms polynomial $f(x) = a_{t-1}x^{t-1} + \cdots + a_1 x + s$.
4. For $1 \le i \le m$ Zelda gives A_i $f(i)$ mod p.

We Used Polynomials. Could Use...

What did we use about degree $t-1$ polynomials?

1. t points determine the polynomial (we need constant term).
2. $t-1$ points give **no information** about constant term.

Could do geometry over $\mathbb{Z}_p^3$. A **Plane** in $\mathbb{Z}_p^3$ is:

$$\{(x,y,z) : ax+by+cz=d\}$$

1. 3 points in $\mathbb{Z}_p^3$ determine a plane.
2. 2 points in $\mathbb{Z}_p^3$ give **no information** about d.

This approach is due to George Blakely, **Safeguarding Cryptographic Keys**, **International Workshop on Managing Requirements**, **Vol 48**, **1979**.
We will not do secret sharing this way, though one could.

We Used Polynomials. Could Use...

We won't go into details but there are two ways to use the **Chinese Remainder Theorem** to do Secret Sharing.

Due to:
C.A. Asmuth and J. Bloom. **A modular approach to key safeguarding. IEEE Transactions on Information Theory Vol 29**, **Number 2**, **208-210**, **1983**.

And Independently

M. Mignotte **How to share a secret**, **Cryptography: Proceedings of the Workshop on Cryptography, Burg Deursetein**, **Volume 149 of Lecture Notes in Computer Science**, **1982**.

Features and Caveats of Poly Method

Imagine that you've done (t, m) secret sharing with polynomial, $p(x)$. So for $1 \leq i \leq m$, A_i has $f(i)$.

1. **Feature:** If more people come FINE- can extend to $(t, m + a)$ by giving A_{m+1}, $f(m + 1)$, ..., A_{m+a}, $f(m + a)$.
2. **Caveat:** If $m > p$ then you run out of points to give people. There are ways to deal with this, but we will not bother. We will always assume $m < p$.